

# Scalar Fields in General Relativity

## Mukhanov and Winitzki Chap 5 Summary

Robert D. Klauber, April 13, 2016 (copyright 2016 ©) [www.quantumfieldtheory.info](http://www.quantumfieldtheory.info)

Ref: Mukhanov, V.F., and Winitzki, S., *Introduction to Quantum Effects in Gravity*, Cambridge (2007), Chap. 5

	<u>Minimally Coupled</u>		<u>Non-Minimally Coupled Example</u>		<u>Comment</u>
	<u>Real Scalar</u>	<u>Gravity</u>	<u>Real Scalar</u>	<u>Gravity</u>	
Field	$\phi$	$g_{\mu\nu}$	$\phi$	$g_{\mu\nu}$	
Back-ground	Physical length (measured in meters with meter sticks) in $x_1$ direction is $dx_1 = \sqrt{-g_{11}} dx_1$ Coordinate length $= dx_1$ (The length of this distance in meters measured with meter sticks depends on the particular generalized coordinate grid chosen, i.e., on $g_{11}$ , which represents that grid in the $x_1$ direction.) Physical 4D volume in an orthogonal (i.e., $g_{\mu\nu}$ is diagonal) spacetime coordinate system is $dV = dx_1 dx_2 dx_3 dx_0$ $= \sqrt{-g_{11}} dx_1 \sqrt{-g_{22}} dx_2 \sqrt{-g_{33}} dx_3 \sqrt{g_{00}} dx_0 = \sqrt{-g} dx_1 dx_2 dx_3 dx_0 = \sqrt{-g} dV = \sqrt{-g} d^4x$				$g_{11}$ is negative; phys value has “~” underneath  $g$ is Det $g_{\mu\nu}$ ; $dV$ is coord volume
Procedure	In standard QFT 1) $\eta_{\mu\nu} \rightarrow g_{\mu\nu}$ 2) “,” $\rightarrow$ “;” 3) $d^4x \rightarrow \sqrt{-g} d^4x$	Choose $\mathcal{L}_{grav}$ as in classical GR	In this example, as in 2 <sup>nd</sup> column, but add extra term.	In this example, $\mathcal{L}_{grav}$ of 3 <sup>rd</sup> column.	In QFT we change to natural units, where $c=\hbar=1$
Action S	$S_m = \int \underbrace{\mathcal{L}_m}_{\text{phys Lagr}} \underbrace{\sqrt{-g} d^4x}_{\text{phys } dV}$ $= \int \underbrace{\mathcal{L}_m}_{\text{coord coord Lagr}} \underbrace{d^4x}_{\text{coord } dV}$	$S_{grav} = \int \underbrace{\mathcal{L}_{grav}}_{\text{phys Lagr}} \underbrace{\sqrt{-g} d^4x}_{\text{phys } dV}$ $= \int \mathcal{L}_{grav} d^4x$	As in 2 <sup>nd</sup> column	As in 3 <sup>rd</sup> column	$\sqrt{-g}$ of 3) above included in $\mathcal{L}_m$ and $\mathcal{L}_{grav}$
Lagrangian $\mathcal{L}$	$\mathcal{L}_m = \sqrt{-g} \left( \frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - V(\phi) \right)$	$\mathcal{L}_{grav} = -\frac{\sqrt{-g}}{8\pi G} (R + 2\Lambda)$	$\mathcal{L}_m = \sqrt{-g} \left( \frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - V(\phi) - \frac{\xi}{2} R \phi^2 \right)$	In this example, $\mathcal{L}_{grav}$ of 3 <sup>rd</sup> column	Other non-min $\mathcal{L}_m$ , $\mathcal{L}_{grav}$ could have other terms in $R_{\alpha\beta\gamma\delta}$
Free field V	$V(\phi) = \frac{1}{2} m^2 \phi^2$		As in 2 <sup>nd</sup> column		“Free”, but $\phi$ , grav coupled
Total $\mathcal{L}$	$\mathcal{L} = \mathcal{L}_m + \mathcal{L}_{grav}$	$\mathcal{L} = \mathcal{L}_m + \mathcal{L}_{grav}$	As in 2 <sup>nd</sup> column	As in 3 <sup>rd</sup> column	
Variation of S	From $\delta S = 0$ , where $\mathcal{L}_{grav}$ has terms in $g_{\alpha\beta,\mu\nu}$ due to $R$ , one gets the Euler-Lagrange eqs below. Gravity eq has an extra term beyond the more familiar Euler-Lagrange eq due to 2 <sup>nd</sup> derivative terms in $\mathcal{L}_{grav}$ . Note the math derivation is based on the integrand used with $d^4x$ , so it uses $\mathcal{L} = \mathcal{L} \sqrt{-g}$ .				
Euler-Lagrange equation	$\partial_\mu \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} - \frac{\partial \mathcal{L}}{\partial \phi} = 0$	$-\partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial g_{\alpha\beta,\mu\nu}}$ $+ \partial_\mu \frac{\partial \mathcal{L}}{\partial g_{\alpha\beta,\mu}} - \frac{\partial \mathcal{L}}{\partial g_{\alpha\beta}} = 0$	As in 2 <sup>nd</sup> column	As in 3 <sup>rd</sup> column	From $\delta S = 0$
Note:	Derivation to get gravity result below is laborious. See Mukhanov & Minitzki pgs. 229-232.				

Equation of motion	$(\sqrt{-g} g^{\alpha\beta} \phi_{,\beta})_{,\alpha} + \sqrt{-g} \frac{\partial V}{\partial \phi} = 0$	$G_{\alpha\beta} + g_{\alpha\beta} \Lambda = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + g_{\alpha\beta} \Lambda = 8\pi G T_{\alpha\beta}$	$(\sqrt{-g} g^{\alpha\beta} \phi_{,\beta})_{,\alpha} + \sqrt{-g} \left( \frac{\partial V}{\partial \phi} + \xi R \phi \right) = 0$	As in 3 <sup>rd</sup> column w different $T_{\alpha\beta}$	From Euler-Lagrange equation
Stress-energy part of above		$T_{\alpha\beta} = \frac{2}{\sqrt{-g}} \frac{\partial \mathcal{L}_m}{\partial g^{\alpha\beta}} = \phi_{,\alpha} \phi_{,\beta} - g_{\alpha\beta} \left( \frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - V(\phi) \right)$		$T_{\alpha\beta} = \frac{2}{\sqrt{-g}} \times \left( \partial_\mu \partial_\nu \frac{\partial \mathcal{L}_m}{\partial g_{\alpha\beta}{}^{\mu\nu}} - \partial_\mu \frac{\partial \mathcal{L}_m}{\partial g^{\alpha\beta}{}_{,\mu}} + \frac{\partial \mathcal{L}_m}{\partial g^{\alpha\beta}} \right)$	$\leftarrow \mathcal{L}_m$ in 5 <sup>th</sup> column from “Lagrangian $\mathcal{L}$ ” row above, 4 <sup>th</sup> column
Covariant form of eq of motion	$\phi_{;\alpha}{}^\alpha + \frac{\partial V}{\partial \phi} = 0$	$G_{\alpha\beta} + g_{\alpha\beta} \Lambda = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + g_{\alpha\beta} \Lambda = 8\pi G T_{\alpha\beta}$	$\phi_{;\alpha}{}^\alpha + \frac{\partial V}{\partial \phi} + \xi R \phi = 0$	As in 3 <sup>rd</sup> column w different $T_{\alpha\beta}$	Gravity field eq of motion above already covariant
Above are	GR version of Klein-Gordon eq	Einstein’s field equation			
	<b><u>E/m Interactions with (Complex) Scalar Field</u></b>				
Procedure	Initially as above with free field, but now need complex scalar field, as real scalar field is charge neutral.				
Action $S$	$S_m = \int \mathcal{L}_m d^4x$	All gravity as above in free scalar field			
Free field Lagrangian $\mathcal{L}$	$\mathcal{L}_m = \sqrt{-g} \times \left( \frac{1}{2} g^{\mu\nu} \phi^*_{,\mu} \phi_{,\nu} - \frac{1}{2} m^2 \phi^* \phi \right)$				For free field $V(\phi^*, \phi) = \frac{1}{2} m^2 \phi^* \phi$
Interaction $\mathcal{L}_m$	with $\phi_{,\mu} \rightarrow D_\mu \phi$ $\mathcal{L}_m = \sqrt{-g} \times \left( \frac{1}{2} g^{\mu\nu} (D_\mu \phi)^* D_\nu \phi - \frac{1}{2} m^2 \phi^* \phi \right)$				$D_\mu = \partial_\mu + iA_\mu$
Re-write interact $\mathcal{L}_m$	$\mathcal{L}_m = \sqrt{-g} \left( \frac{1}{2} g^{\mu\nu} \phi^*_{,\mu} \phi_{,\nu} - \frac{1}{2} m^2 \phi^* \phi - \frac{1}{2} g^{\alpha\beta} i A_\alpha (\phi^* \phi_{,\beta} - \phi \phi^*_{,\beta}) + \frac{1}{2} g^{\alpha\beta} A_\alpha A_\beta \phi^* \phi \right)$				
Interaction $\mathcal{L}_{e/m}$	$F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ Standard QFT $\mathcal{L}_{e/m} = -\frac{1}{16\pi} \eta^{\alpha\beta} \eta^{\mu\nu} F_{\alpha\mu} F_{\beta\nu} \rightarrow -\frac{1}{16\pi} g^{\alpha\beta} g^{\mu\nu} F_{\alpha\mu} F_{\beta\nu}$				$\Gamma_{\beta\gamma}^\alpha$ drop out of $F_{\mu\nu}$
Total $\mathcal{L}$	$\mathcal{L} = \mathcal{L}_m + \mathcal{L}_{grav} + \mathcal{L}_{e/m}$				
Equations of Motion	Use above $\mathcal{L}$ in Euler-Lagrange eq for 3 fields, $\phi$ , $g_{\mu\nu}$ , and $A_\mu$ . Get 3 coupled eqs.				